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# Transient fractal behavior in snapshot attractors of driven chaotic systems

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Snapshot attractors, i.e., attractors formed by a cloud of trajectories at the same instants of time, are usually employed to reveal the fractal structure of randomly or chaotically driven dynamical systems. A necessary condition for the underlying fractal structure to be observed is that the ensemble of particles utilized in the construction of the snapshot attractors are subject to identical perturbation at any instant of time. We examine the influence of small phase-space inhomogeneity in the chaotic perturbation on the observability of the snapshot fractal attractors. We find that, typically, fractal structure can be seen in only a transient period of time. The scaling of the transient time with the amount of inhomogeneity is investigated. Implication to experimental observation of fractal structure in physical systems is pointed out. [S1063-651X(99)02308-9]

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## I. INTRODUCTION

The concept of *snapshot attractors* was first proposed by Romeiras, Grebogi, and Ott [1] to study the fractal structure of chaotic attractors in randomly or chaotically driven dynamical systems. Consider a physical system that exhibits a chaotic attractor. In a noiseless situation, the attractor typically exhibits a fractal structure that is caused by the underlying chaotic dynamics. Such a fractal structure can be visualized in the phase space if the chaotic system is low-dimensional, that is, at each point along a trajectory on the attractor, there is a stable and an unstable direction. The boundedness of the phase-space region in which the attractor lies, together with an exponential *divergence of distances along the unstable* direction, stipulates that distances be folded back on the unstable manifold, thus forming a fractal set of foliations in the stable direction [2,3]. Under the influence of small random perturbations, the fractal pattern moves randomly in the phase space from time to time. As such, if one examines a long trajectory produced by the dynamics, one usually observes that the fractal structure is smeared out to a distance scale that is proportional to the strength of the perturbations. In order to see the fractal structure of the underlying chaotic attractor, a remedy is to ''freeze'' the time and examine the snapshot patterns formed by an ensemble of trajectories. Starting with a cloud of uniformly distributed initial conditions, after an initial transient time, one can indeed see the fractal structure of the snapshot attractors [1,4]. The details of the fractal structure differ from time to time, but properties such as fractal dimensions remain invariant [1], although such invariant properties may fluctuate slightly about their nominal values in practical situations [5]. The idea of snapshot attractors has also been used in laboratory experiments to visualize and investigate fractal patterns arising in physical situations such as passive particles convected on the surface of fluid [6]. More recently, snapshot attractors were utilized to study the transition to chaos in quasiperiodically driven dynamical systems [7].

A necessary condition for snapshot attractors to exhibit fractal structures is that at any instant of time, the influence of the random perturbation on *every* trajectory in the cloud must be identical. The reason is that the difference in the random perturbation to different trajectories in the cloud can be regarded as a phase-space diffusion, which, when it is large enough, can smear out the fractal structure even in snapshot attractors. To be more specific, consider low-dimensional driven chaotic systems described by the following map:

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,\mathbf{y}_n), \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^{N_x}$ and $y\in\mathbb{R}^{N_y}$ represent random or chaotic driving. Now imagine that as time progresses, we move the fractal pattern within a phase-space region in a random fashion. In order to see the fractal structure, we must move the *entire* pattern at any instant of time. That is, every point on the attractor must be shifted by an identical amount to preserve the fractal pattern. Thus, the random perturbations $\mathbf{y}_n$ must not depend on the phase-space variable $\mathbf{x}_n$ [1,4]. It they do, the fractal structure of the snapshot attractor will be smeared approximately by an amount proportional to the magnitude of the perturbation. Due to chaos in the driving system, the fractal pattern will be less and less visible because the amount of ''fuzziness'' in the phase space is magnified exponentially in time. As such, it is not possible to observe a fractal structure even in the snapshot attractor for a long time.

The aim of this paper is to address to what extent fractal snapshot attractors can be observed in dynamical systems driven by random or chaotic perturbations that depends only weakly on the dynamical variables. We call such weak dependence *phase-space inhomogeneity*. Specifically, we consider dynamical systems described by Eq. (1) and assume that the chaotic driving signal comes from the following process:

$$\mathbf{y}_{n+1}=\mathbf{G}(\mathbf{y}_n,\epsilon\mathbf{x}_n), \tag{2}$$

where $\mathbf{G}$ is a nonlinear map and $\epsilon\gtrsim 0$ represents the amount of weak phase-space inhomogeneity. Note that when $\epsilon=0$, the $\mathbf{x}$ dynamics do not influence the $\mathbf{y}$ dynamics and, hence, Eqs. (1) and (2) represent a *unidirectionally coupled* (from $\mathbf{y}$ to $\mathbf{x}$) system. Suppose we choose an ensemble of initial conditions $\mathbf{x}_0$ and evolve them according to Eqs. (1) and (2).

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If $\epsilon=0$, then at any instant of time $n$, the perturbations to every trajectory in the ensemble are identical. In this case, a snapshot image of all these trajectory points would reveal a fractal structure [1]. If, however, $\epsilon \gtrsim 0$, this phase-space inhomogeneity will be amplified exponentially due to the chaotic nature of the driving (2), and finite scale fractal structures can be seen in the snapshot attractors *for only a transient period of time*. We find that the average transient time $\tau$ is typically short, and it scales with $\epsilon$ as

$$\tau \approx \alpha \ln \frac{1}{\epsilon} + \beta, \tag{3}$$

where the proportional constant $\alpha$ is approximately the inverse of the largest Lyapunov exponent of Eq. (2). The implication of the scaling relation (3) is that in order to observe fractal snapshot attractors for a long time, the phase-space inhomogeneity must be small and/or the driving system is only weakly chaotic. For unidirectionally coupled systems where $\epsilon=0$, Eq. (3) gives $\tau=\infty$, so fractal patterns in snapshot attractors can be observed indefinitely.

## II. NUMERICAL RESULTS

Consider the following two-dimensional Ikeda-Hammel-Jones-Moloney map [8]:

$$\begin{aligned} x_{n+1} &= a + b(x_n \cos\phi_n - y_n \sin\phi_n), \\ y_{n+1} &= b(x_n \sin\phi_n + y_n \ \phi_n), \end{aligned} \tag{4}$$

with the following phase variable $\phi_n$:

$$\phi_n = k - p/(1 + x_n^2 + y_n^2) + 2\pi\theta_n, \tag{5}$$

where $\theta_n$ represents the random or chaotic driving and $a$, $b$, $k$, and $p$ are parameters. The Ikeda-Hammel-Jones-Moloney map models the dynamics of an optical pulse propagating in a ring cavity, subject to partial reflection, phase and amplitude modulation and distortion due to a nonlinear optical medium in the cavity. Specifically, the optical field is represented by the complex variable $z=x+iy$, the parameter $a$ and $b$ quantify the splitting of the optical field at various mirrors in the cavity, the term $p/(1+x_n^2+y_n^2)$ simulates the phase modulation due to the nonlinear medium, and the parameter $k$ characterizes the optical detuning of the cavity in the absence of a nonlinear medium. The random perturbation $\theta_n$ can be regarded as coming from the temporal fluctuations of the nonlinear optical medium, or from the interaction with another chaotic optical cavity. To model the effect of phase-space inhomogeneity in the driving, we assume that $\theta$ comes from the following chaotic logistic map [9]:

$$\theta_{n+1} = 3.75\theta_n(1-\theta_n) + \epsilon x_n. \tag{6}$$

Note that Eqs. (4) and (6) are actually a system of two bidirectionally coupled nonidentical chaotic maps. We choose $(a,b,k,p)=(0.85, 0.9, 0.4, 5.18)$, so the Ikeda-Hammel-Jones-Moloney map, in the absence of perturbation $\theta_n$, exhibits a chaotic attractor, which apparently has a fractal structure [8].

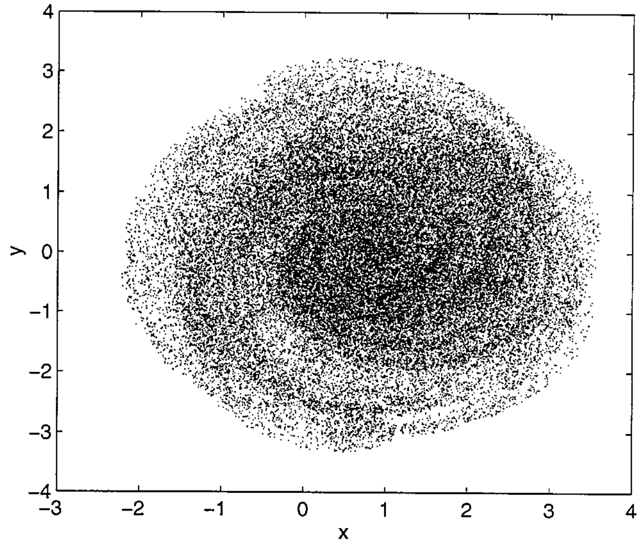


FIG. 1. For the Ikeda-Hammel-Jones-Moloney map, a smeared chaotic attractor from a single trajectory of 50 000 points under the influence of small random noise.

We first examine the case of unidirectional coupling where $\epsilon=0$ so that the $(x,y)$ dynamics in Eq. (4) does not influence the $\theta$ dynamics in Eq. (6). Due to the coupling to the $(x,y)$ dynamics from the $\theta$ dynamics, the fractal structure in the chaotic attractor of the Ikeda-Hammel-Jones-Moloney map is smeared if a single long trajectory is examined, as shown in Fig. 1. But since there is no coupling from the $(x,y)$ dynamics to the $\theta$ dynamics (no phase-space inhomogeneity), the underlying fractal geometry of the chaotic attractor in Eq. (4) can still be revealed by observing snapshot attractors, as shown in Figs. 2(a)–2(d) at time $n=1000$, 2000, 3000, and 4000, respectively. To obtain Figs. 2(a)–2(d), we choose a grid of $128\times128$ initial conditions uniformly distributed in the phase-space region: $(-2.0\leqslant x\leqslant4.0,\ -2.5<y<2.5)$ and evolve all these initial conditions according to Eqs. (4), (5), and (6) under the condition $\epsilon=0$. We see that for unidirectionally coupled systems, the fractal pattern in the snapshot attractors can indeed be observed for an arbitrarily long time, and it is known that snapshot attractors possess the same multifractal geometry as that of the chaotic attractor in the absence of random perturbations [1].

We now examine the effect of weak phase-space inhomogeneity on snapshot attractors. Figures 3(a)–3(j) show, for $\epsilon=10^{-16}$, the snapshot attractors from the same set of initial conditions used to obtain Figs. 2(a)–2(d), at times $n=10,20,...,100$, respectively. We see that the snapshot attractors are apparently fractal for $20\lesssim n\lesssim80$, beyond which time the fractal structure is smeared out. The apparently nonfractal behavior at very short time, e.g. at $n=10$, is due to the fact that it takes a finite amount of time for the cloud of trajectories to settle down to the chaotic attractor. As $\epsilon$ is increased, the time interval for fractal snapshot attractors to be observed decreases, as shown in Figs. 4(a)–4(j), where $\epsilon=10^{-10}$. To measure the average transient time interval $\tau$ in which snapshot attractors are apparently fractal, we use the following box-counting procedure. We divide the phase-space region from which the initial conditions are chosen into a grid of $200\times200$ boxes. We then count, at each instant

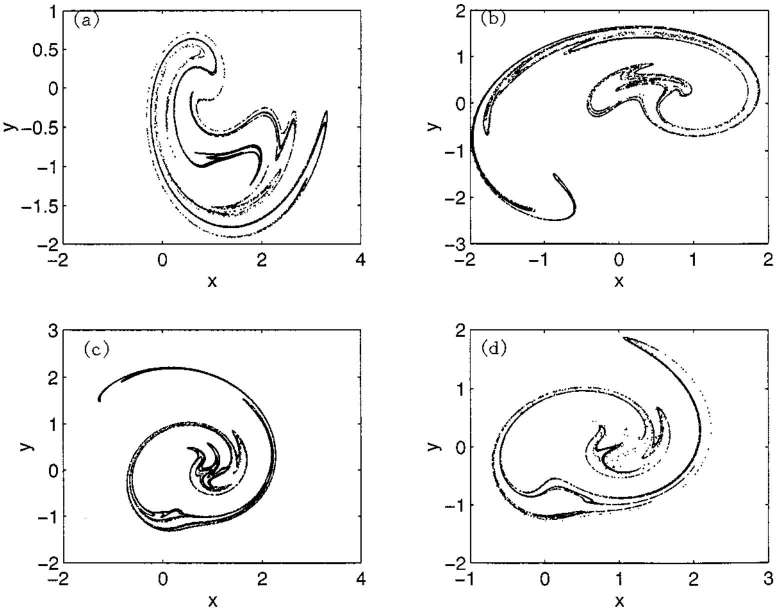


FIG. 2. Fractal snapshot attractors observed in the Ikeda-Hammel-Jones-Moloney map at time $n=1000$ (a), $n=2000$ (b), $n=3000$ (c), and $n=4000$ (d), where $\epsilon=0$.

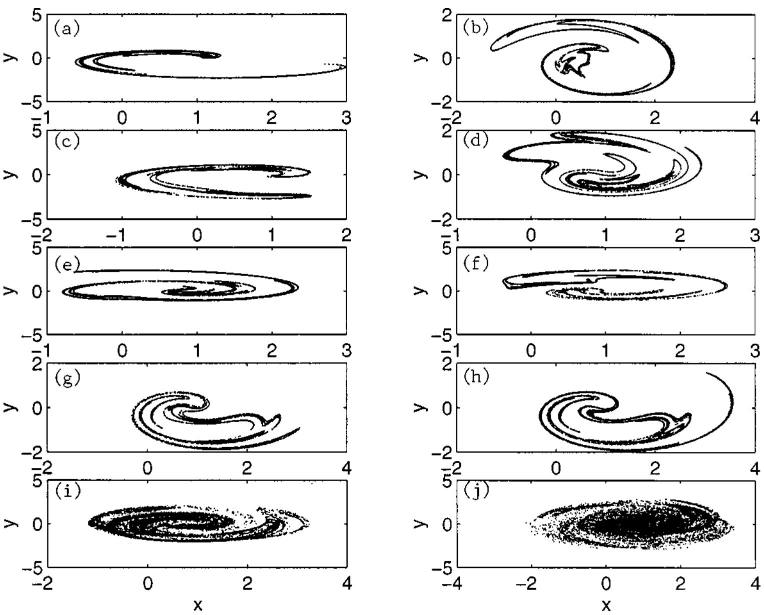


FIG. 3. For the chaotically driven Ikeda-Hammel-Jones-Moloney map with phase-space inhomogeneity $\epsilon=10^{-16}$, the snapshot attractors at times $n=10,20,...,100$ (a)–(j).

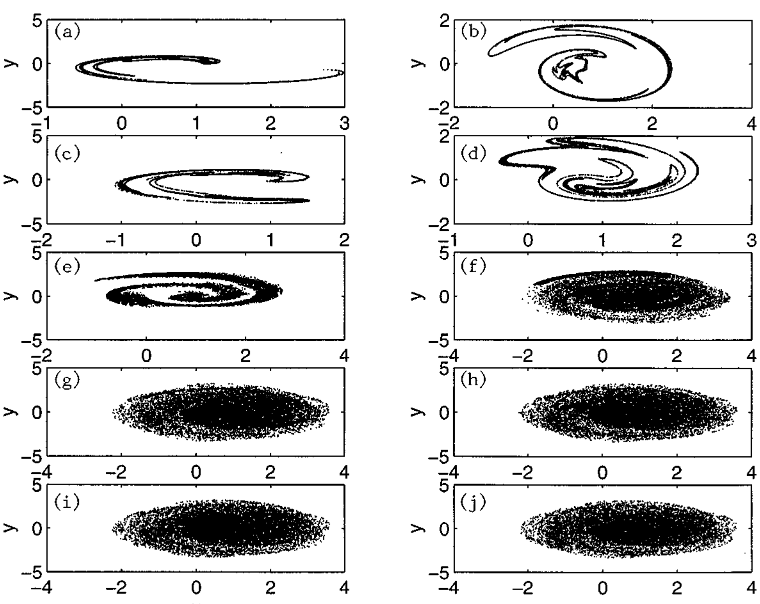


FIG. 4. For the chaotically driven Ikeda-Hammel-Jones-Moloney map with phase-space inhomogeneity $\epsilon=10^{-10}$, the snapshot attractors at times $n=10,20,...,100$ (a)–(j).

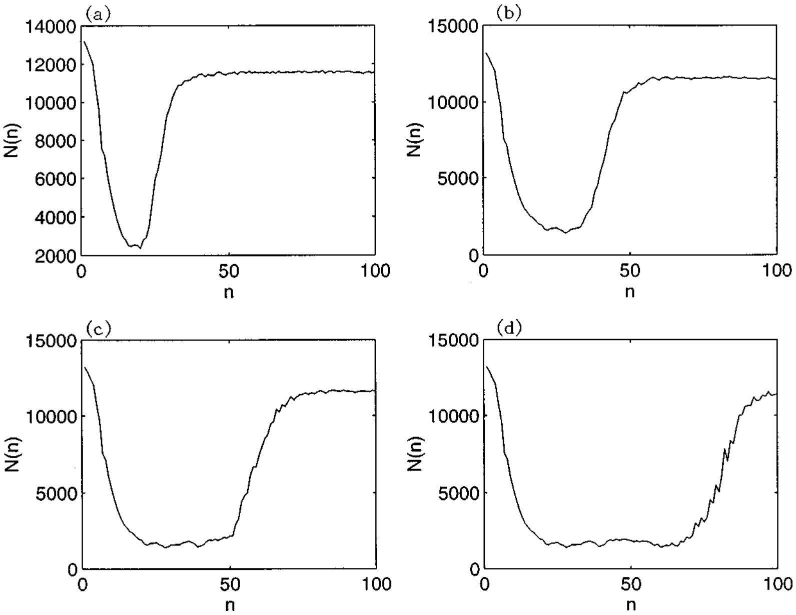


FIG. 5. (a)–(d) The number of nonempty boxes $N(t)$ versus $t$ for $\epsilon=10^{-6}$, $10^{-9}$, $10^{-12}$, and $10^{-15}$, respectively.

of time $n$, the number of nonempty boxes $N(t)$. For small time, since the trajectories have not come close to the chaotic attractor, we expect to observe a large number of occupied boxes. As the trajectories begin to settle down in the vicinity of the chaotic attractor, $N(t)$ starts to decrease and reach a small value and remains approximately at this value when the snapshot attractors are apparently fractal. When the effect of phase-space inhomogeneity in the driving begins to take over so that the fractal structure becomes smeared, we expect the number of nonempty boxes to increase. The time interval in which $N(t)$ remains approximately at constant is taken to be the average transient time $\tau$. Figures 5(a)–5(d) show $N(t)$ versus $t$ for $\epsilon=10^{-6}$, $10^{-9}$, $10^{-12}$, and $10^{-15}$, respectively. We see that compared with the order of magnitude of decrease in $\epsilon$, the transient time $\tau$ only increases incrementally. Figure 6 shows $\tau$ versus $\log_{10}\epsilon$, where we observe the scaling relation (3). The slope in the plot is approximately $-5.75$, which gives $\alpha\approx 5.75/\ln 10\approx 2.5$. We notice that the Lyapunov exponent of Eq. (6) is about 0.36, the inverse of which is approximately 2.78. This agrees reasonably well with the result in Fig. 6. These features appear to be general, regardless of the specific choice of the chaotic map or the chaotic driving [10].

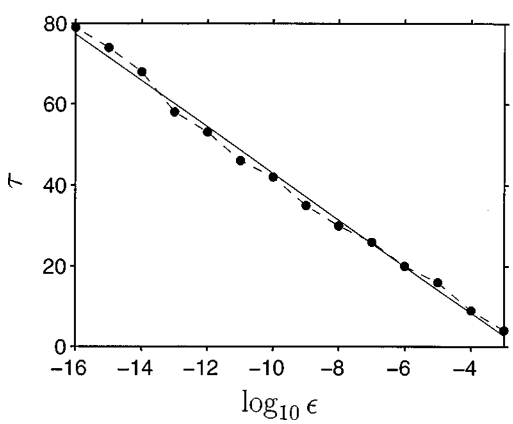


FIG. 6. The average transient time $\tau$ versus $\log_{10}\epsilon$.

## III. HEURISTIC ARGUMENT FOR THE SCALING RELATION (3)

We now give a heuristic argument for the scaling behavior observed in Fig. 6. To do so, it is necessary to define the Lyapunov exponents of both Eqs. (1) and (2). Since, however, Eqs. (1) and (2) are coupled together, their Lyapunov exponents are in fact the *sub-Lyapunov exponents* that were originally introduced in the context of chaos synchronization [11]. Let $\lambda_i^x$ $(i=1,...,N_x)$ and $\lambda_j^y$ $(j=1,...,N_y)$ be the Lyapunov exponents of the subsystems Eqs. (1) and (2), respectively. Mathematically, the exponents can be defined as follows:

$$\lambda_i^x=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln\left|\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\bigg|_{(\mathbf{x}_n,\mathbf{y}_n)}\cdot\mathbf{u}_i\right|,\quad i=1,...,N_x$$

$$\lambda_j^y=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln\left|\frac{\partial \mathbf{G}}{\partial \mathbf{y}}\bigg|_{(\mathbf{x}_n,\mathbf{y}_n)}\cdot\mathbf{v}_j\right|,\quad j=1,...,N_y \tag{7}$$

where $(\partial\mathbf{F}/\partial\mathbf{x})|_{(\mathbf{x}_n,\mathbf{y}_n)}$ and $(\partial\mathbf{G}/\partial\mathbf{y})|_{(\mathbf{x}_n,\mathbf{y}_n)}$ are the Jacobian matrices of Eqs. (1) and (2) evaluated *along a coupled trajectory* $(\mathbf{x}_n,\mathbf{y}_n)$, $\mathbf{u}_i$ $(i=1,...,N_x)$ is a unit vector in the $i$th eigendirection in the tangent space of Eq. (1), and $\mathbf{v}_j$ $(j=1,...,N_y)$ is a unit vector in the $j$th eigendirection in the tangent space of Eq. (2). These vectors can be obtained by using the standard Gran-Schmit orthogonalization procedure [12].

Say we choose a cloud of initial conditions uniformly distributed in a phase-space region covering the attractor of the subsystem $\mathbf{x}$. Let $\delta$ be the smallest distance scale to resolve the fractal structure in an observation. Roughly, the time $T_x$ for the cloud to settle down to a fractal set of resolution $\delta$ can be estimated from $e^{-|\lambda_{N_x}^x|T_x}\sim\delta$. We obtain, $T_x\sim-\ln\delta/|\lambda_{N_x}^x|$. In order to be able to observe the fractal structure, the amount of phase-space inhomogeneity $\epsilon$ must be smaller than $\delta$. The time $T_y$ for diffusion to reach the distance scale $\delta$ in the $\mathbf{x}$ subsystem satisfies $\epsilon e^{\lambda_1^y T_y} \sim \delta$. We obtain $T_y \sim (\ln \delta - \ln \epsilon)/\lambda_1^y$. Thus, the time window for a fractal snapshot attractor to be observed is

$$\tau = T_y - T_x \sim \frac{-\ln \epsilon}{\lambda_1^y} + \ln \delta \left( \frac{1}{\lambda_1^y} + \frac{1}{|\lambda_{N_x}^x|} \right), \qquad (8)$$

which gives the scaling relation (3). Demanding $\tau > 0$, we obtain the maximum value of the phase-space inhomogeneity for a fractal snapshot attractor of resolution $\delta$ to be observed: $\epsilon < \delta e^{-D}$, where $D \equiv 1 + \lambda_1^y / |\lambda_{N_x}^x|$.

## IV. DISCUSSIONS

We remark that the phenomenon of transient fractal snapshot attractors has some implications to the study of fractal geometry in high-dimensional chaotic systems, i.e., systems with more than one positive Lyapunov exponents. Consider the following general class of high-dimensional systems,

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}(\mathbf{x}_n, \epsilon_x \mathbf{y}_n), \\ \mathbf{y}_{n+1} &= \mathbf{g}(\mathbf{y}_n, \epsilon_y \mathbf{x}_n), \end{aligned} \qquad (9)$$

where both **f** and **g** are chaotic maps and $\epsilon_x$ and $\epsilon_y$ are two parameters characterizing the coupling from **x** to **y** and vice versa. The system setting of Eq. (9) arises naturally in the context of coupled chaotic oscillators, which has become an area of intense recent interest [13]. The maps **f** and **g** can be *noninvertible* [14]. In order to study the fractal geometry of system Eq. (9), we assume that when the two maps are uncoupled, i.e., when $\epsilon_x = \epsilon_y = 0$, both maps $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{y})$ exhibit a chaotic attractor with one positive Lyapunov exponent and the attractor has a fractal structure in their own phase space **x** and **y**. When couplings are present, the coupling terms $\epsilon_x \mathbf{y}$ and $\epsilon_y \mathbf{x}$ can be regarded as two driving terms to the **x** and **y** dynamics, respectively. Since **y** and **x** are chaotic variables, the problem effectively becomes that of studying fractals of randomly driven chaotic systems. Intuitively we expect snapshot attractors in the **x** or **y** space to reveal the fractal structures in the absence of couplings. Nonetheless, due to coupling, the influence of driving is not homogeneous in both the **x** and **y** subspaces. The phase-space inhomogeneity of the chaotic driving thus becomes a potential obstacle for observing low-dimensional fractal structures in high-dimensional chaotic systems. The fact that fractal snapshot attractors have been observed in laboratory experiments such as passive particles convected on the surface of fluids [6] indicates that the experimental condition may be such that the amount of phase-space inhomogeneity in the fluid surface is near zero (the coupling between the dynamics in the direction orthogonal to the fluid surface and the dynamics of the passive scalar on the surface of the fluid is nearly unidirectional) or, the dynamics of the driving is only weakly chaotic with a near zero positive largest Lyapunov exponent.

## ACKNOWLEDGMENTS

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